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Investigation of Geometrically Nonlinear Vibrations of Laminated Shallow Shells with Layers of Variable Thickness by Meshless Approach

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Abstract
Geometrically nonlinear vibrations of laminated shallow shells with layers of variable thickness are studied. Nonlinear equations of motion for shells based on the first order shear deformation and classical shells theories are considered. In order to solve this problem we use the numerically-analytical method proposed in work [1]. Accordingly to this approach the initial problem is reduced to consequences of some linear problems including linear vibrations problem, special elasticity ones and nonlinear system of ordinary differential equations in time. The linear problems are solved by the variational Ritz' method and Bubnov-Galerkin procedure combined with the R-functions theory [2]. To construct the basic functions that satisfy all boundary conditions in case of simply-supported shells we propose new solutions structures. The proposed method is used to solve both test problems and new ones.

Keywords
R-functions method, laminated shallow shells, nonlinear vibration, variable thickness

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Introduction

Extensive literature reviews on nonlinear vibrations of plates and open shallow shells have been given by many scientists [1-3]. A huge number of publications is devoted to this issue. But virtually no studies that have investigated multilayer shallow shells with layers of varying thickness. In this paper we attempt to develop an algorithm for solving this class of problems. Proposed algorithm applies meshless discretization. It is based on combination of the classical approaches and modern constructive tools of the R-functions theory [8]. Application of R-functions theory allows studying geometrically nonlinear dynamic response of the laminated shallow shells and plates with complex shape and different boundary conditions. We present also new types of structure formulas that allow to construct appropriate system of basic functions. These basic functions satisfy exactly all boundary conditions.

1. Mathematical Formulation

Laminated shallow shells of an arbitrary plan form with radii of curvature R_x, R_y which consist of M layers of the variable thickness $h_i(x, y)$ are considered. Assume that the plane of $x_1 O x_2$ coincides with the mid-surface of the shallow shell. The shell theories used in the present investigation are shear deformable theory (FSDT) and the classical shell theory (CST). According to these theories we assume that the tangent displacements are linear functions of coordinate z and the transverse displacement w is a constant along the thickness of the shell. Let us recall that the CST adopts Kirchhoff's hypothesis. But FSDT is based on hypothesis of straight line. This means that the normal to the mid-surface remains straight line after deformation, but not necessary normal to the mid surface. In the abbreviated form the nonlinear stress strain relations can be written as follows:

$$\{F\}=[A]\cdot\{\varepsilon\} \tag{1}$$

where

$$\{F\}=\{N_{11},N_{22},N_{12},M_{11},M_{22},M_{12},Q_x,Q_y\},\ [A]=\begin{bmatrix}[C] & [K] & 0\\ [K] & [D] & 0\\ 0 & 0 & [S]\end{bmatrix}$$

$$\{\varepsilon\}=\{\varepsilon_{11},\varepsilon_{22},\varepsilon_{12},\chi_{11},\chi_{22},\chi_{12},\varepsilon_{23},\varepsilon_{13}\},\ \varepsilon_{11}=u,_x+\frac{w}{R_x}+\frac{1}{2}w,_x^2,\ \varepsilon_{22}=v,_y+\frac{w}{R_y}+\frac{1}{2}w,_y^2$$

$$\varepsilon_{12}=u,_y+v,_x+w,_x\,w,_y,\ \varepsilon_{13}=\delta\left(w,_x+\psi_x-\frac{u}{R_x}\right),\ \varepsilon_{23}=\delta\left(w,_y+\psi_y-\frac{v}{R_y}\right),$$

$$\chi_{11}=\delta\psi_x,_x-(1-\delta)w_{,xx},\ \chi_{22}=\delta\psi_y,_y-(1-\delta)w,_{yy},\ \chi_{12}=\delta(\psi_x,_y+\psi_y,_x)-2(1-\delta)w,_{xy}$$

Here u, v and w are the displacements at the mid-surface, ψ_x and ψ_y are the rotations about the y- and x-axes respectively and N, M, Q are the stress, moment and the transverse shear resultants. Matrices $[C]$, $[D]$ and $[K]$ have the following form:

$$[C]=\begin{bmatrix}C_{11} & C_{12} & C_{16}\\ C_{12} & C_{22} & C_{26}\\ C_{16} & C_{26} & C_{66}\end{bmatrix},\ [D]=\begin{bmatrix}D_{11} & D_{12} & D_{16}\\ D_{12} & D_{22} & D_{26}\\ D_{16} & D_{26} & D_{66}\end{bmatrix}$$

$$[S]=\begin{bmatrix}S_{44} & S_{45}\\ S_{54} & S_{55}\end{bmatrix},\ [K]=\begin{bmatrix}K_{11} & K_{12} & K_{16}\\ K_{12} & K_{22} & K_{26}\\ K_{16} & K_{26} & K_{66}\end{bmatrix}$$

Since the laminate consists of a number of variable thickness lamina, the elements of the constitutive matrices $[A]$, $[C]$, $[K]$, $[D]$, $[S]$ are expressed as [2,7]:

$$\begin{gathered}\left(C_{ij}(x,y),K_{ij}(x,y),D_{ij}(x,y)\right)=\sum_{m=1}^{M}\int_{h_m(x,y)}^{h_{m+1}(x,y)}B_{ij}^{(m)}\left(1,z,z^2\right)dz,\quad (i,j=1,2,6)\\ S_{ij}(x,y)=k_i^2\sum_{m=1}^{M}\int_{h_m(x,y)}^{h_{m+1}(x,y)}B_{ij}^{(m)}dz,\quad (i,j=4,5)\end{gathered} \tag{2}$$

In the expressions (2) values $B_{ij}^{(m)}$ are stiffness coefficients of the m-th layer; k_i ($i=\overline{4,5}$) are shear correction factors. Further we assume that $k_4=k_5=5/6$, that is $S_{45}=S_{54}$. Indicator δ is the tracing constant which takes values 1 and 0 for the FSDT and CST respectively. It should be noted that problem about nonlinear vibrations of shallow shells with symmetric layers is essentially simple than relative problem for nonsymmetrical layers. This is explained by the fact that factors K_{ij} vanish and matrix A takes the form:

$$[A]=\begin{bmatrix}[C] & 0 & 0\\ 0 & [D] & 0\\ 0 & 0 & [S]\end{bmatrix}$$

In this paper we will only consider symmetric cross-ply and angle-ply laminated shallow shells.

2. Method of solution

We will apply the method proposed in works [7]. According to this approach the first step is study of linear problem in order to find the natural frequencies and eigenfunctions $\{U^{(c)}\}=\{u^{(c)},v^{(c)},w^{(c)},\psi_x^{(c)},\psi_y^{(c)}\}^T$ that satisfy the given boundary conditions. Solution of linear problems for laminated shells with variable thickness we will fulfill by RFM [8]. Note that we will not ignore inertia forces solving linear problem. Since linear vibrations are harmonic ones, then this problem may be reduced to variational problem about finding minimum of the following functional

$$J=\Pi_{\max}-T_{\max} \tag{3}$$

where $\Pi_{\max}$, $T_{\max}$ are strain and kinetic energies relatively. These energies are defined by the following expressions:

$$U_{\max}=\frac{1}{2}\iint_{\Omega}\left[N_{11}\varepsilon_{11}+N_{22}\varepsilon_{22}+N_{12}\varepsilon_{12}+M_{11}\chi_{11}+M_{22}\chi_{22}+M_{12}\chi_{12}+\delta\left(Q_x\varepsilon_{13}+Q_y\varepsilon_{23}\right)\right]d\Omega$$

$$T_{\max}=\frac{\Omega_L^2}{2}\iint_{\Omega}\left(\rho(x,y)\left(h(x,y)\left(u^2+v^2+w^2\right)+\delta\frac{h^3(x,y)}{12}\left(\psi_x^2+\psi_y^2\right)\right)\right)d\Omega$$

In order to find extreme of the functional (3) we will use method by Ritz. The system of basic functions we will build by R-functions theory. That is why first we construct the corresponding solutions structures [6-8], which satisfy the given boundary conditions.

3. Solutions structures for different boundary conditions

Below we present solutions structure for some boundary conditions for case of the classical theory.

Clamped edge. Solution structure is known for this case and may be found in references [6-8]. Solution structures have more bulky form for other boundary conditions. Below we present some of them.

Movable simply supported edge. Let us consider the following boundary conditions:

$$w=0,\quad M_n=0 \tag{4}$$

$$N_n=0,\quad T_n=0 \tag{5}$$

It is possible to prove that the following structures satisfy all boundary conditions (4-5):

$$v=\Phi_2-\omega D_1\Phi_2-\frac{\omega P}{P^2+\omega^2}\left[P_2^{(v)}T_1\Phi_2-P_1^{(v)}T_1\Phi_1\right]$$

$$u=\Phi_1-\omega D_1\Phi_1-\frac{\omega P}{P^2+\omega^2}\left[P_1^{(u)}T_1\Phi_1+P_2^{(u)}T_1\Phi_2\right]$$

$$w=\omega\Phi_3-\frac{\omega^2 B_1}{2(B_1^2+\omega^2)}\left(B_1\left(2D_1\Phi_3+\Phi_3 D_2\omega\right)+2B_2\left(\Phi_3 D_1 T_1\omega+T_1\Phi_3\right)+B_3\Phi_3 T_2\omega\right)$$

Here

$$P_1^{(u)} = C_2 P_3 - C_3 P_2\,,\; P_2^{(u)} = C_4 P_3 - C_3 P_4$$

$$P_1^{(v)} = P_1 C_2 - C_1 P_2\,,\; P_2^{(v)} = C_1 P_4 - C_4 P_1\,,\; P = C_1 P_3 - C_3 P_1$$

$$C_1 = C_{11}(\partial_x\omega)^3 + 3C_{16}(\partial_x\omega)^2(\partial_y\omega) + (C_{12}+2C_{66})(\partial_x\omega)(\partial_y\omega)^2 + C_{26}(\partial_y\omega)^3$$

$$C_2 = C_{16}(\partial_x\omega)^3 + (2C_{66}-C_{11})(\partial_x\omega)^2(\partial_y\omega) + (C_{26}-2C_{16})(\partial_x\omega)(\partial_y\omega)^2 - C_{12}(\partial_y\omega)^3$$

$$C_3 = C_{16}(\partial_x\omega)^3 + (C_{12}+2C_{66})(\partial_x\omega)^2(\partial_y\omega) + 3C_{26}(\partial_x\omega)(\partial_y\omega)^2 + C_{22}(\partial_y\omega)^3$$

$$C_4 = C_{12}(\partial_x\omega)^3 + (2C_{26}-C_{16})(\partial_x\omega)^2(\partial_y\omega) + (C_{12}-2C_{66})(\partial_x\omega)(\partial_y\omega)^2 - C_{26}(\partial_y\omega)^3$$

$$P_1 = C_{16}(\partial_x\omega)^3 + (C_{11}+C_{66}-C_{12})(\partial_x\omega)^2(\partial_y\omega) - C_{26}(\partial_x\omega)(\partial_y\omega)^2 - C_{66}(\partial_y\omega)^3$$

$$P_2 = C_{66}(\partial_x\omega)^3 - C_{26}(\partial_x\omega)^2(\partial_y\omega) - (C_{11}+C_{66}-C_{12})(\partial_x\omega)(\partial_y\omega)^2 + C_{16}(\partial_y\omega)^3$$

$$P_3 = C_{66}(\partial_x\omega)^3 + C_{16}(\partial_x\omega)^2(\partial_y\omega) + (C_{12}-C_{22}-C_{66})(\partial_x\omega)(\partial_y\omega)^2 - C_{26}(\partial_y\omega)^3$$

$$P_4 = C_{26}(\partial_x\omega)^3 + (C_{12}-C_{22}-C_{66})(\partial_x\omega)^2(\partial_y\omega) - C_{16}(\partial_x\omega)(\partial_x\omega)^2 + C_{66}(\partial_x\omega)^3$$

$$B_1 = -D_{11}(\partial_x\omega)^4 - 4D_{16}(\partial_x\omega)^3(\partial_y\omega) - 2(D_{12}+2D_{66})(\partial_x\omega)^2(\partial_y\omega)^2 -$$
$$-4D_{26}(\partial_x\omega)(\partial_y\omega)^3 - D_{22}(\partial_y\omega)^4$$

$$B_2 = -D_{16}(\partial_x\omega)^4 + (D_{11}-D_{12}-2D_{66})(\partial_x\omega)^3(\partial_y\omega) + 3(D_{16}-D_{26})(\partial_x\omega)^2(\partial_y\omega)^2 +$$
$$+(D_{12}-D_{22}+2D_{26})(\partial_x\omega)(\partial_y\omega)^3 + D_{26}(\partial_y\omega)^4$$

$$B_3 = -D_{12}(\partial_x\omega)^4 + 2(D_{16}-D_{26})(\partial_x\omega)^3(\partial_y\omega) - (D_{11}+D_{22}-4D_{66})(\partial_x\omega)^2(\partial_y\omega)^2 -$$
$$-2(D_{16}-D_{26})(\partial_x\omega)(\partial_y\omega)^3 - D_{12}(\partial_y\omega)^4$$

The function $\omega(x,y)$ satisfies the following conditions:

$$\omega(x,y) \succ 0, \forall (x,y)\in\Omega, \qquad \omega(x,y)\big|_{\partial\Omega} = 0$$

Hence $\omega(x,y)=0$ is equation of the domain boundary. To build this function we will apply R-functions theory worked out by V.L.Rvachev [8].

The differential operators $D_m f(x,y), T_m f(x,y)$ are defined as [8]:

$$D_m f(x,y) = (\nabla\omega,\nabla)^m f = (\partial_x\omega\cdot\partial_x + \partial_y\omega\cdot\partial_y)^m f\,,\quad T_m f(x,y) = (\partial_x\omega\cdot\partial_y - \partial_y\omega\cdot\partial_x)^m f$$

Note that boundary condition will be satisfied at any choice of the indefinite components Φ_1, Φ_2, Φ_3.

Immovable simply supported edge. The boundary conditions of the immovable simply supported edge are described as

$$u=0,\quad v=0,\quad w=0,\quad M_n=0 \tag{6}$$

We can prove that the following structures of solution satisfy boundary conditions (6)

$$u=\omega\Phi_1\,,\qquad v=\omega\Phi_2\,,$$

$$w=\omega\Phi_3 - -\frac{\omega^2 B_1}{2(B_1^2+\omega^2)}\left(B_1(2D_1\Phi_3+\Phi_3 D_2\omega) + 2B_2(\Phi_3 D_1 T_1\omega + T_1\Phi_3) + B_3\Phi_3 T_2\omega\right)$$

In order to obtain the basic functions we will expand the indefinite components in series on some complete system of functions (power or trigonometric polynomial, splines or others).

Like previous case we can obtain appropriate structures for case of the first order of the shear deformable theory.

4. Method of solving nonlinear problem

Let us present unknown functions in the following form:

$$w=\sum_{i=1}^{n} y_i(t) w_i^{(c)}(x,y), \quad \psi_x=\delta\sum_{i=1}^{n} y_i(t)\psi_{xi}^{(c)}(x,y), \quad \psi_y=\delta\sum_{i=1}^{n} y_i(t)\psi_{yi}^{(c)}(x,y)$$

$$u=\sum_{i=1}^{n} y_i(t) u_i^{(c)}(x,y)+\sum_{i=1}^{n}\sum_{j=1}^{n} y_i y_j u_{ij}, \quad v=\sum_{i=1}^{n} y_i(t) v_i^{(c)}(x,y)+\sum_{i=1}^{n}\sum_{j=1}^{n} y_i y_j v_{ij} \tag{7}$$

where $y_k(t)$ are unknown functions in time, $w_i^{(c)}(x,y), u_i^{(c)}(x,y), v_i^{(c)}(x,y), \psi_{xi}^{(c)}(x,y), \psi_{yi}^{(c)}(x,y)$ are components of the *i*-th eigenfunctions of linear vibration problem. Functions u_{ij}, v_{ij} must be solutions of the following system [6-7]

$$\begin{cases} L_{11}u_{ij}+L_{12}v_{ij}=-Nl_1^{(2)}(w_i,w_j) \\ L_{21}u_{ij}+L_{22}v_{ij}=-Nl_2^{(2)}(w_i,w_j) \end{cases}$$

Obtained system is solved by RFM. Substituting the expressions (7) for functions u, v, w, ψ_x, ψ_y in initial system of equation of motion and applying procedure by Bubnov-Galerkin we get nonlinear system of ordinary differential equations in unknown functions $y_j(t)$:

$$y_j''(t)+\alpha_j y_j(t)+\sum_{i=1}^{n}\sum_{k=1}^{n}\beta_{jik} y_i(t) y_k(t)+\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\gamma_{jikl} y_i(t) y_k(t) y_l(t)=\tilde{F} \quad \left(j=\overline{1,n}\right) \tag{8}$$

Expressions for coefficients $\alpha_j, \beta_{jik}, \gamma_{jikl}$ are found and expressed through double integrals of known functions. In order to solve the obtained system (8) we will apply method by Runge-Kutta.

5. Numerical results

The developed approach is validated on some tested problems and will be applied to solve new ones.

Problem. Consider three-layers clamped shallow shell with square planform of side a and thickness $h=0.01a$. Suppose that the face layers are isotropic, but middle layer is orthotropic with the following mechanical constants:

$$E_1/E_0=0.25,\ E_2/E_0=0.077,\ G_{12}/E_0=0.029,\ \nu_1=0.24.$$

Here E_0 is elastic modulus for isotropic layers, Poisson's ratio for isotropic layers $\nu_0=0.3$ and density of all layers is taken by the same $\rho=\rho_0$. As middle surface we take the plane $z=0$. Assume that thickness of layers varies linearly, but the general thickness is a constant and defined as:

$$\sum_{s=1}^{3} h_s = h.$$

Equations of surfaces which bound the inner layer maybe written as (Fig.1):

$$z=\frac{h}{2}\left(m+\frac{x}{a}(1-2m)\right),\quad z=-\frac{h}{2}\left(m+\frac{x}{a}(1-2m)\right)$$

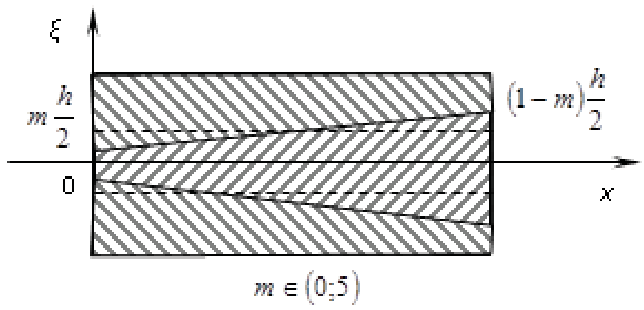

Figure 1. Surface bounding the inner layer

In the given case rigid coefficients are expressed by the following relations:

$$C_{ij}(x,y)=h\left(E_0+(E_1-E_0)\left(m+\frac{x}{a}(1-2m)\right)\right),$$

$$D_{ij}(x,y)=\frac{h^3}{12}\left(E_0+(E_1-E_0)\left(m+\frac{x}{a}(1-2m)\right)^3\right)$$

Table 1. Comparison of the values for three layers square clamped plate

λ_i	**Meth.**	$m=0$	$m=0.25$	$m=0.5$
λ_1	RFM	0.886	1.023	1.057
	[4]	0.88	1.02	1.06
λ_2	RFM	3.608	4.259	4.369
	[4]	3.60	4.25	4.35
λ_3	RFM	3.781	4.264	4.429
	[4]	3.80	4.26	4.40

The values of non-dimensional frequencies $\lambda_i=\omega_i^2a^2\rho_0/(E_oh^2),(i=1,2,3)$ obtained by proposed method are presented in the Table 1. In paper [4] the similar results were obtained for plates. Comparison of received results with available confirms the validation of proposed method. In the Tables 2 we present values of non-dimensional frequencies $\lambda_i=\omega_ia^2\sqrt{\rho_0/E_0h^2}$ for cylindrical and spherical shells with square planform.

Table2. Effect of parameter *m* on values of non-dimensional frequencies of the clamped cylindrical and spherical shells

Cylindrical shells($k_1=0.25$, $k_2=0$)				**Spherical shells ($k_1=0.25$, $k_2=0$ .25)**		
λ_i	$m=0$	$m=0.25$	$m=0.5$	$m=0$	$m=0.25$	$m=0.5$
λ_1	18.287 2	19.5 27	19.836	24.382	26.836	27.562
λ_2	20.633	22.0 42	22.392	26.941	28.918	28.992
λ_3	25.275	26.9 96	27.382	28.023	29.078	29.565
λ_4	30.983	32.8 13	33.269	34.422	36.524	37.027

The backbone curves were also obtained for clamped and simply supported spherical and cylindrical panels.

6. Conclusion

Numerically-analytical approach for investigation of nonlinear vibration of shallow shells with layers of variable thickness is developed. New solution structures satisfying all boundary conditions corresponding movable and immovable simply supported edge are proposed for shells with symmetrical layers. The present approach has advantage of being suitable for considering different types of the boundary conditions in domains of arbitrary shape.

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